Preserved Curvature conditions along the RF

for more applications in dim 3 we proceed as follows.

Let $\{e_i\}$ be an o.n. frame on $U \subseteq M^n$.

$\rightsquigarrow$ we get an o.n. basis $\{\theta^k = \theta^k_{ij}\, e_i \wedge e_j\}$ of $\Lambda^2 TU$.

$\Rightarrow \forall x \in U$, $\varphi_x : \Lambda^2 T_x M^n \longrightarrow \Lambda^2 \mathbb{R}^n$ given by

$$(\theta^1, \ldots, \theta^N) \longmapsto (\beta_1, \ldots, \beta_N) \qquad N = {}^nC_2$$

an ordered basis $\longmapsto$ ordered basis is a well-defined

Lie algebra homomorphism.

$\because M^3$ is parallelizable $\Rightarrow$ we have a global frame $\{e_i\}$ so

we get an o.n. basis $\{\theta^k = \theta^k_{ij}\, e_i \wedge e_j\}$ of $\Lambda^2 TM^3$.

e.g. such a basis can be taken to be

$$\theta^1 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3) \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$\theta^2 = \frac{1}{\sqrt{2}}(e_3 \wedge e_1) \sim \begin{pmatrix} 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \end{pmatrix}$$

$$\theta_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2) \sim \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One can also calculate the Lie algebra square by noticing that $\langle [\theta^i, \theta^j], \theta^k \rangle$ is fully alternating in (i,j,k) $\Rightarrow$

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}^{\#} = \begin{bmatrix} df-e^2 & ce-bf & be-cd \\ ce-bf & af-c^2 & bc-ae \\ be-cd & bc-ae & ad-b^2 \end{bmatrix}$$

We use the same symbol Rm to identify the quadratic form Rm on $\Lambda^2 TM^3$, i.e.,

$$Rm(e_i \wedge e_j, e_k \wedge e_l) = \langle Rm(e_i, e_j)e_k, e_l \rangle.$$

and using the basis $\{\theta^1, \theta^2, \theta^3\}$ to identify Rm as a 3×3 matrix at every point.

$$\langle Rm(e_i, e_j)e_k, e_l \rangle = (Rm)_{pq}\, \theta^p_{ij}\, \theta^q_{kl}$$

$\therefore$ If $\{e_i\}$ evolves to remain orthonormal then by the Uhlenbeck's trick we know Rm satisfies the PDE

$$\partial_t Rm = \Delta Rm + Rm^2 + Rm^{\#}$$

and hence its behaviour is governed by the ODE

$$\frac{d}{dt} Rm = Rm^2 + Rm^{\#} \text{ in each fiber.}$$

Choose $\{e_i\}$ so that $Rm(0)$ is diagonal at $x \in M^3$ w/ eigenvectors $\lambda(0) \geq \mu(0) \geq \nu(0)$ then we get the corresponding ODE as

$$\frac{d}{dt}\begin{pmatrix} \lambda(t) & & 0 \\ & \mu(t) & \\ 0 & & \nu(t) \end{pmatrix} = \begin{pmatrix} \lambda^2 & & 0 \\ & \mu^2 & \\ 0 & & \nu^2 \end{pmatrix} + \begin{pmatrix} \mu\nu & & 0 \\ & \lambda\nu & \\ 0 & & \lambda\mu \end{pmatrix}$$

$\Rightarrow$ $Rm(t)$ remains diagonal $\forall t$ and

$$\left.\begin{aligned} \frac{d}{dt}\lambda &= \lambda^2 + \mu\nu \\ \frac{d}{dt}\mu &= \mu^2 + \lambda\nu \\ \frac{d}{dt}\nu &= \nu^2 + \lambda\mu. \end{aligned}\right\} \text{——} \; (1).$$

<u>Claim</u> :- $\lambda(t) \geq \mu(t) \geq \nu(t)$ as long as the sol^n exists.

note that
$$\frac{d}{dt}(\lambda - \mu) = (\lambda - \mu)(\lambda + \mu - \nu)$$
$$\frac{d}{dt}(\mu - \nu) = (\mu - \nu)(-\lambda + \mu + \nu)$$

$$\Rightarrow \quad \frac{d}{dt} \log(\lambda - \mu) = \lambda + \mu - \nu$$

$$\frac{d}{dt} \log(\mu - \nu) = -\lambda + \mu + \nu$$

$$\Rightarrow \quad \log(\lambda(t) - \mu(t)) = \log\left(\lambda(0) - \mu(0)\right) + \int_0^T (\lambda + \mu - \nu)\, dt$$

$$\log(\mu(t) - \nu(t)) = \log\left(\mu(0) - \nu(0)\right) + \int_0^T (-\lambda + \mu + \nu)\, dt$$

If e.g., $\mu(t) - \nu(t) > 0$ then we are done. If at some t_0 we get $\mu(t_0) = \nu(t_0)$ then $\log(\mu - \nu)$ is defined on $[0, t_0)$ and $\to -\infty$ as $t \nearrow t_0$.

$$\Rightarrow \quad \lim_{t \to t_0} (-\lambda + \mu + \nu) = -\infty \Rightarrow \text{atleast one of } \lambda, \mu \text{ or } \nu \text{ has}$$

a discontinuity at t_0 and they cease to exist.

$\therefore$ as long as the sol^n exist $\boxed{\lambda(t) \geq \mu(t) \text{ and } \mu(t) \geq \nu(t).}$

Applications:-

1) Let $C_0 \in \mathbb{R}$ and let $K = \{ M \mid \lambda + \mu + \nu \geq C_0 \}$

$\because$ trace of M is a linear function $\Rightarrow$ convex. By the criterion for invariance under parallel translation, we get that K is invariant under parallel translation.

$\therefore$ we want to check that the ODE $\frac{d}{dt} M = M^2 + M^{\#}$ is preserved by K.

$$\frac{d}{dt}(\lambda+\mu+\nu) = \lambda^2+\mu^2+\nu^2+\lambda\mu+\mu\nu+\lambda\nu$$

$$= \frac{1}{2}\left[(\lambda+\mu)^2+(\mu+\nu)^2+(\lambda+\nu)^2\right]$$

$$\geq \frac{2}{3}(\lambda+\mu+\nu)^2 \geq 0.$$

$\therefore$ if $Rm \in K$ at $t=0$ then it remains so $\forall t$.

This is just another way of saying that if $R \geq C_0$ at $t=0$ then $R \geq C_0 \; \forall t$ which we already knew before.

Remark :- we are, of course, using the fact that in dim 3, in terms of the eigenvalues of Rm,

$$Ric = \frac{1}{2}\begin{pmatrix} \mu+\nu & & 0 \\ & \lambda+\nu & \\ 0 & & \lambda+\mu \end{pmatrix} \text{ and } R = \lambda+\mu+\nu.$$

② Let $K = \{ M \mid \nu(M) \geq 0 \}$. ν is the lowest eigenvalue of M

and the map $\nu : V_x \to \mathbb{R}$ is a concave function b/c

$\because$ $\nu(M) = \min\limits_{|v|=1} M(v,v)$

$\therefore$ $\nu(sM_1 + (1-s)M_2) = \min\limits_{|v|=1}\left[s M_1(v,v) + (1-s) M_2(v,v)\right]$

$$\geq \min_{|v|=1} s\, M_1(v,v) + \min_{|v|=1} (1-s)\, M_2(v,v)$$

$$= s\,\nu(M_1) + (1-s)\,\nu(M_2)$$

$\Rightarrow$ ν is a concave function

$\Rightarrow$ the set $\mathcal{K}$ of M w/ $\nu(M) \geq 0$ is a convex function.

∞ $\nu(sM_1 + (1-s)M_2) \geq 0 \Rightarrow sM_1 + (1-s_2)M_2 \in \mathcal{K}$ if $M_1, M_2 \in \mathcal{K}$.

what about the preservation of the ODE?

$$\frac{d}{dt}\nu = \nu^2 + \lambda\mu$$

if either $\nu(0) > 0 \Rightarrow \frac{d\nu}{dt} > 0 \Rightarrow \nu(t) > 0$.

If $\nu(0) = 0$ and $\mu(0) > 0 \Rightarrow \frac{d\nu}{dt}(0) = \nu(0)^2 + \lambda(0)\mu(0) > 0$

$\Rightarrow \nu(t) > \nu(0) > 0 \;\forall\; t > 0$.

If ν, μ and λ are all initially 0 then they remain so, so again $\nu(t) \geq 0$.

If $\nu(0) = \mu(0) = 0$ and $\lambda(0) > 0$ then $\nu(t)$ and $\mu(t)$ remain $= 0$

but $\frac{d}{dt}\lambda = \lambda^2 \Rightarrow \lambda(t) > 0$.

In any case K is preserved by ODE $\Rightarrow$ if $Rm(0) \in K$ then $Rm(t) \in K \ \forall t$

$\Rightarrow$ $Rm \geq 0$ is preserved in dim 3.

③ (Exercise) $K = \{ M \mid \mu(M) + \nu(M) \geq 0 \}$ $\Rightarrow$ $Rc \geq 0$ is preserved.

Sol^n same idea as before $\Rightarrow$ $\mu + \nu$ is a concave function $\Rightarrow$ K is a convex set. for preservation in the ODE

$$\frac{d}{dt}(\nu + \mu) = \nu^2 + \mu^2 + \lambda\mu + \lambda\nu = \mu^2 + \nu^2 + \lambda(\mu + \nu) \geq 0$$

whenever $\mu + \nu \geq 0$.

$\Rightarrow$ K is preserved by the ODE. $\Rightarrow$ $Rc \geq 0$ is preserved.

④ [Ricci pinching is preserved, i.e. if the eigenvalues of Rm are initially close together then they remain so $\forall t$] Exe:- check that K is convex.

If $\lambda(Rm) \leq C(\mu(Rm) + \nu(Rm))$ for $C \geq 1/2$ then it remain so.

Let $K = \{ M \mid \lambda(M) \leq C(\mu(M) + \nu(M))$ for a given $C \geq 1/2 \}$.

$$\left.\begin{array}{l} \text{If } C = 1/2 \Rightarrow \lambda(M) \leq \frac{1}{2}(\mu(M) + \nu(M)). \\ \text{But } \because \lambda \geq \mu,\ \lambda \geq \nu \Rightarrow \lambda \geq \frac{1}{2}(\mu + \nu) \end{array}\right\} \Rightarrow \lambda = \frac{\mu + \nu}{2}.$$

Also, $\lambda \geq \mu \Rightarrow \frac{\lambda+\nu}{2} \geq \frac{\mu+\nu}{2}$

$\therefore \frac{\lambda+\nu}{2} \geq \lambda$

$\Rightarrow \frac{\nu}{2} \geq \frac{\lambda}{2} \Rightarrow \lambda = \mu = \nu.$

$\therefore \lambda(t) = \nu(t) = \mu(t) \ \forall t \Rightarrow$ the ODE is preserved.

Assume now, $C > \frac{1}{2}$ and $\lambda(0) \geq \mu(0) \geq \nu(0)$

then we must have $\mu(0) + \nu(0) \geq 0$ b/c $\because \lambda(0) \geq \frac{1}{2}(\mu(0) + \nu(0))$

If $\mu(0) + \nu(0) < 0$ then we can never have $\lambda(0) \leq C(\mu(0) + \nu(0))$

w/ $C > \frac{1}{2}$. So we must have $\mu(0) + \nu(0) \geq 0$.

now, look at $\frac{d}{dt}(\mu+\nu) = \mu^2 + \nu^2 + \lambda(\mu+\nu)$.

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here i) either $\mu(0) = \nu(0) = \lambda(0) = 0 \Rightarrow \mu(t) = \nu(t) = \lambda(t) = 0$

and the ODE trivially preserves the set K.

ii) Or it can happen that $\mu(0) + \nu(0) > 0 \Rightarrow \frac{d}{dt}(\mu+\nu) > 0$

$\Rightarrow \mu(t) + \nu(t) > 0 \ \forall t$.

$\Rightarrow \lambda(t) \geq \frac{\mu(t) + \nu(t)}{2} > 0$ so we can take logarithms.

we get $\frac{d}{dt} \log\left(\frac{\lambda}{\mu+\nu}\right) = \frac{\mu+\nu}{\lambda} \frac{d}{dt}\left(\frac{\lambda}{\mu+\nu}\right)$

$$= \frac{1}{\lambda(\mu+\nu)} \left[(\mu+\nu)\frac{d}{dt}\lambda - \lambda \frac{d}{dt}(\mu+\nu) \right]$$

$$= \frac{\lambda}{\lambda(\mu+\nu)} \left((\mu+\nu)(\lambda^2+\mu\nu) - \lambda(\nu^2+\lambda\mu+\mu^2+\lambda\nu) \right)$$

$$= \frac{1}{\lambda(\mu+\nu)} \left[\cancel{\lambda^2\mu} + \mu^2\nu + \cancel{\lambda^2\nu} + \mu\nu^2 - \lambda\nu^2 - \cancel{\lambda^2\mu} - \lambda\mu^2 - \cancel{\lambda^2\nu} \right]$$

$$= \frac{\mu^2(\nu-\lambda) + \nu^2(\mu-\lambda)}{\lambda(\mu+\nu)} \leq 0$$

$$\therefore \quad \frac{\lambda(t)}{\mu(t)+\nu(t)} \leq \frac{\lambda(0)}{\mu(0)+\nu(0)} \leq C \quad w/\ C \geq \tfrac{1}{2} \text{ is preserved}$$

by the ODE.

$\therefore$ if the initial eigenvalues of Rm at a point are pinched then they remain so.

Recall in dim 3, above estimate says that

$$\frac{Rg}{3} \leq \lambda(Rm)g \leq C\,Rc \implies Rc \geq \frac{Rg}{3C} = \varepsilon Rg$$

w/ $\varepsilon = \frac{1}{3C}$ and $\therefore$ we get a better estimate.

Remark :- If $Ric(g(0)) > 0$ then —— estimate is satisfied and $\therefore$ we get the Ricci pinching estimate for some $C < \infty$.

5) Ricci pinching is improved, i.e., the metric $g(t)$ is almost Einstein at points where the scalar curvature is very large.

Let $C_0 > 0$, $C_1 \geq \frac{1}{2}$, $C_2 < \infty$ and $0 < \delta < 1$. Consider

$$K = \left\{ M \;\middle|\; \begin{array}{l} \lambda + \mu + \nu \geq C_0 \\ \lambda \leq C_1(\mu+\nu) \\ \lambda - \nu - C_2(\lambda+\mu+\nu)^{1-\delta} \leq 0 \end{array} \right\}$$

Exe:- show that K is a convex set.

The first two inequalities are already preserved by the ODE. so we look at the third one.

note that:- the 1^{st} and the 2^{nd} ineq $\Rightarrow$ $\mu+\nu > 0$ as if $\mu+\nu \leq 0$ then $C_1(\mu+\nu) \leq 0 \Rightarrow \lambda \leq 0 \Rightarrow \lambda+\mu+\nu \leq 0$ which is not possible. $\therefore$ $\mu+\nu > 0$ (this also implies that $Ric(g(0)) > 0$).

$\because$ $\lambda - \nu > 0$ and so is $\lambda+\mu+\nu$ $\Rightarrow$ $\log\left(\dfrac{\lambda-\nu}{(\lambda+\mu+\nu)^{1-\delta}}\right)$ makes sense.

$$\therefore \frac{d}{dt}\log\left(\frac{\lambda-\nu}{(\lambda+\mu+\nu)^{1-\delta}}\right) =$$

$$\frac{(\lambda+\mu+\nu)^{1-\delta}}{\lambda-\nu}\left[\frac{(\lambda+\mu+\nu)^{1-\delta}\frac{d}{dt}(\lambda-\nu) - (\lambda-\nu)(1-\delta)(\lambda+\mu+\nu)^{-\delta}\frac{d}{dt}(\lambda+\mu+\nu)}{(\lambda+\mu+\nu)^{2-2\delta}}\right]$$

$$= \frac{[\lambda+\mu+\nu]^{1-\delta}}{\lambda-\nu}\left[\frac{(\lambda+\mu+\nu)^{1-\delta}(\lambda^2+\mu\nu-\nu^2-\lambda\mu) - (\lambda-\nu)(1-\delta)(\lambda+\mu+\nu)^{-\delta}[\lambda^2+\mu\nu+\mu^2+\lambda\nu+\nu^2+\lambda\mu]}{(\lambda+\mu+\nu)^{2-2\delta}}\right]$$

$$= \frac{[\lambda+\mu+\nu]^{-1}}{\lambda-\nu}\left[(\lambda+\mu+\nu)(\lambda^2+\mu\nu-\nu^2-\lambda\mu) - (1-\delta)(\lambda-\nu)(\lambda^2+\mu\nu + \mu^2+\lambda\nu+\nu^2+\lambda\mu)\right]$$

$$= \frac{[\lambda+\mu+\nu]^{-1}}{\lambda-\nu}\left[(\lambda+\mu+\nu)((\lambda+\nu)(\lambda-\nu)+\mu(\nu-\lambda)) - (1-\delta)(\lambda-\nu)(\nu(\nu+\mu)+\lambda(\mu+\nu) + \lambda^2+\mu^2)\right]$$

$$= [\lambda+\mu+\nu]^{-1}\left[(\lambda+\mu+\nu)(\lambda+\nu-\mu) - (1-\delta)(\lambda^2+\mu^2+(\lambda+\nu)(\mu+\nu))\right]$$

$$= [\lambda+\mu+\nu]^{-1}\left[\lambda^2+\cancel{\lambda\nu}-\cancel{\lambda\mu}+\cancel{\lambda\mu}+\cancel{\mu\nu}-\mu^2+\lambda\nu+\cancel{\nu^2}-\nu\mu - \cancel{\lambda^2}-\mu^2-\lambda\mu-\cancel{\lambda\nu}-\cancel{\nu\mu}-\cancel{\nu^2} + \delta\lambda^2+\delta\mu^2+\delta\lambda\mu+\delta\lambda\nu+\delta\nu\mu+\delta\nu^2\right]$$

$$= [\lambda+\mu+\nu]^{-1}\left[\delta\lambda(\lambda+\nu+\mu)+\delta\nu(\mu+\nu+\lambda)-\delta\nu\lambda - \delta\mu(\lambda+\mu+\nu)+2\delta\mu^2+\delta\mu\lambda+\delta\mu\nu-\mu^2+\lambda\nu-\nu\mu - \mu^2-\lambda\mu\right]$$

$$= \delta(\lambda+\nu-\mu) + [\lambda+\mu+\nu]^{-1}\left[-\delta\nu\lambda+2\delta\mu^2+\delta\mu\lambda+\delta\mu\nu-\mu^2+\lambda\nu - \mu\nu-\mu^2-\lambda\mu\right]$$

final

$$= \delta(\lambda+\nu-\mu) - (1-\delta)\frac{[(\mu+\nu)\mu + (\mu-\nu)\lambda + \mu^2]}{(\lambda+\mu+\nu)}$$

$$\leq \delta(\lambda+\nu-\mu) - \frac{(1-\delta)\mu^2}{(\lambda+\mu+\nu)}$$

(as $\nu+\lambda-\mu \leq \lambda \leq C_1(\mu+\nu)$ by the assumption which is $\leq 2C_1\mu$.

$\Rightarrow \mu > 0 \Rightarrow (\mu+\nu)\mu + (\mu-\nu)\lambda > 0$)

But $\mu+\lambda \leq 2\lambda \leq 2C_1(\mu+\nu)$ by the assumptions and the ordering of the eigenvalues

$$\therefore \frac{\mu^2}{(\lambda+\mu+\nu)} = \frac{\mu\cdot\mu}{\lambda+\mu+\nu} \geq \frac{\mu\cdot\frac{(\mu+\nu)}{2}}{3\lambda} \geq \frac{\mu}{6C_1}$$

moreover, $\lambda+\mu-\nu \leq \lambda \leq C_1(\mu+\nu) \leq 2C_1\mu$

$\therefore$ the above ineq. becomes

$$\frac{d}{dt}\log\left(\frac{\lambda-\nu}{(\lambda+\mu+\nu)^{1-\delta}}\right) \leq 2\delta C_1\mu - \frac{(1-\delta)\mu}{6C_1}$$

which on choosing $12\delta C_1^2 \leq 1-\delta$, i.e., $\delta \leq \frac{1}{1+12C_1^2}$

$$\Rightarrow \frac{d}{dt}\log\left(\frac{\lambda-\nu}{(\lambda+\mu+\nu)^{1-\delta}}\right) \leq 0$$

$$\Rightarrow \quad \lambda-\mu \leq C_2(1+\mu+\nu)^{1-\delta}$$

and hence the ODE is preserved.

Exercise :- Prove that the above is equivalent to

$$|Rc - \frac{1}{3}Rg| \leq C R^{1-\delta}.$$

{ This is telling us that the sectional curvatures get "pinched" together as the curvature explodes.

This follows b/c $\lambda - \nu \geq |Rc - \frac{1}{3}Rg|$.

Thus the estimates we have proved for the curvatures along the RF are

$$Ric \geq \varepsilon R g \qquad \varepsilon \leq \frac{1}{3}$$

and

$$\left|Ric - \frac{1}{3}Rg\right| \leq CR^{1-\delta}.$$

If $[0,T)$ is the maximal existence time of our RF, we know that

$$R_{min}(t) \geq \frac{1}{R_{min}(0)^{-1} - \frac{2}{3}t}$$

and $R_{min}(0) > 0$ when $Ric(g(0)) > 0 \Rightarrow T \leq \frac{3}{2R_{min}(0)} < \infty$.

and the above estimate can be written as

$$\frac{|Ric - \frac{1}{3}Rg|^2}{R^2} \leq CR^{-\delta}$$

w/ the LHS being scale-invariant and RHS $\to 0$ if $R(\cdot t) \to \infty$.